Short-form Analysis of Sportsbook and Casino Arbitrage with Markov Chains

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Abstract

This paper undertakes an analysis on sportsbook and casino promotions. The objective is to establish a methodology that guarantees deterministic returns, with a negligible risk profile.

1 Sportsbook

1.1 Prologue

Odds on sportbook lines are interpreted as decimals, over 'American' style. In other words, a stake on a winning line has an arbitrary multiplier C. The stake is multiplied by C and added to the bankroll. A stake on a losing line is removed from the bankroll. We assume no opportunity for arbitrage, and the existence of mutually exclusive lines; i.e for 2 lines, F and U, where $\mathbb{I}_\ell$ defines that line ℓ has occurred, $\mathbb{I}_F \iff \mathbb{I}_{\perp U}$ and $\mathbb{I}_U \iff \mathbb{I}_{\perp F}$. Accounting for vigorish V, any hedging strategy assumes odds exist in this range:

$$1.00 < \frac{C_F + C_U}{C_F \times C_U} = V \quad | \quad 1.00 \leq C_F \leq 2.00 \leq C_U \leq 7.00 \tag{1}$$

A site credit bet operates like a normal cash bet, with an arbitrary multiplier C on a winning line. A bonus bet does not return its stake on a winning line. Therefore, a bonus bet has a reduced multiplier $C - 1$ on a winning line. Both bets are converted to cash on a winning line. Hedges are always in cash, labelled as h_i, whereas promotional bets are labelled as β_i. Both exist in some variable amount.

1.2 Conversion

This section provides insight on strategy, and what conversion rate one can reasonably expect through the process of promotion churning. In most analysis, cashflows are aggregated to show the net PnL, rather than separating each cashflow.

1.2.1 Site Credit Conversion

Assuming some site credit β_{SC}. A site credit is realized as a positive cashflow in a winning line, and is dissolved in a losing line; to reiterate, there is no cash loss from a lost site credit. The PnL equation is structured as:

$$\text{PnL} = \underset{C_i, C_j, h_0}{\arg\max} \; (\beta_{SC} \times C_i - h_0) \times \mathbb{I}_i + (h_0 \times (C_j - 1)) \times \mathbb{I}_j \tag{2}$$

Taking the arg max with indicator functions is ambiguous. We bridge this gap by finding the intersection, which is a PnL that is invariant of outcome, i.e a form of determinism in profit. The promotional bet follows the favourite line, and the hedge bet follows the underdog line. This minimizes the total necessary liquidity to make this conversion feasible, and to not disrupt further conversions. On the contrary, in cases where vigorish is high, the promotional bet should follow the underdog line.

$$0 = (\beta_{SC} \times C_i - h_0) - (h_0 \times (C_j - 1)) \tag{3}$$

$$h_0 = \beta_{SC} \times \frac{C_i}{C_j} \quad \text{where} \tag{4}$$

$$C_i = C_F \text{ and } C_j = C_U \longrightarrow \min h_0 \tag{5}$$

$$C_i = C_U \text{ and } C_j = C_F \longrightarrow \max \text{Conv } \% \tag{6}$$

The conversion rate is determined as:

$$\text{Conv }\% = \frac{\beta_{SC} \times C_i - h_0}{\beta_{SC}} = \frac{C_i \times (C_j - 1)}{C_j} \tag{7}$$

The range of viable odds from equation 1 maps to the range of conversion rates:

$$85\% \leq \text{Conv }\% \leq 100\% \tag{8}$$

1.2.2 Bonus Bet Conversion

Assume some bonus bet β_{BB}. The procedure is strikingly similar, apart from the multiplier.

$$0 = (\beta_{BB} \times (C_i - 1) - h_0) - (h_0 \times (C_j - 1)) \tag{9}$$

$$h_0 = \beta_{BB} \times \frac{C_i - 1}{C_j} \tag{10}$$

The $(C_i - 1)$ term throws off the conversion rate. Bonus bets don't share the same duality that site credit bets have across lines. We optimize C_i and C_j according to the conversion rate to determine a strategy.

$$\text{Conv }\% = \frac{(C_i - 1) \times (C_j - 1)}{C_j} \quad \text{where } \underset{C_i, C_j}{\arg\max}\ \text{Conv }\% := \{C_i = C_F \text{ and } C_j = C_U\} \tag{11}$$

As a note for bonus bets, the most optimal case follows any lines where $C_F >> C_U$, however these are rare and are left as improbable. We list a realistic range of conversion rates for bonus bets:

$$55\% \leq \text{Conv }\% \leq 76\% \tag{12}$$

1.3 Insured Bets

Sportbooks package promotions in the form of insured bets, where the insurance payoff is in the form of site credit bets or bonus bets of an amount that was staked and lost. This opens us up to 3 possible scenarios:

1. First bet on promotional line wins. Insurance is voided, and we are done.
2. First bet on promotional line loses $\iff$ First hedge wins $\longrightarrow$ Second bet on promotional line wins, staked through insurance payoff.
3. First bet on promotional line loses $\iff$ First hedge wins $\longrightarrow$ Second hedge wins.

For case scenarios 2 and 3, we have 2 separate hedges. The first hedge and promotional bet is staked on the event with lines $E_0 = \{i, j\}$, and the second hedge and promotional bet is staked on a separate event $E_1 = \{m, n\}$. Oppositely, case scenario 1 does not rely on E_1, since an insurance payoff can only be exercised when a bet on the promotional line is lost. Also note for the first hedge and promotional bet, both bets are in the form of cash. The site credit or bonus bet is only provided when the cash-based promotional bet is lost for the first time.

1.3.1 Site Credit

Assume the insurance payoff is in the form of a site credit, valued at a stake β that was lost during E_0. Further assume the promotional side takes lines $\{i, m\}$ whereas the hedge side takes lines $\{j, n\}$. We structure profit equations for all 3 scenarios as:

$$\begin{bmatrix} C_i - 1 & -1 & 0 \\ C_m - 1 & C_j - 1 & -1 \\ -1 & C_j - 1 & C_n - 1 \end{bmatrix} \times \begin{bmatrix} \beta \\ h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} \text{PnL}_1 \\ \text{PnL}_2 \\ \text{PnL}_3 \end{bmatrix}$$

For an outcome-invariant profit, we search for $[h_0, h_1]$ constrained by $\text{PnL}_1 = \text{PnL}_2 = \text{PnL}_3 = \text{PnL}$. We rewrite this as $Ax = y$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 - C_j & 1 \\ 1 & 1 - C_j & 1 - C_n \end{bmatrix} \times \begin{bmatrix} \text{PnL} \\ h_0 \\ h_1 \end{bmatrix} = \beta \begin{bmatrix} C_i - 1 \\ C_m - 1 \\ -1 \end{bmatrix}$$

We take the analytical inverse of A, where the determinant exists in every case:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{C_j \times C_n} \times \begin{bmatrix} C_n \times (C_j - 1) & C_n - 1 & 1 \\ C_n & 1 - C_n & -1 \\ 0 & C_j & -C_j \end{bmatrix}$$

As such, we can derive hedge sizes and PnL according to this equation:

$$\begin{bmatrix} \text{PnL} \\ h_0 \\ h_1 \end{bmatrix} = \frac{\beta}{C_j \times C_n} \times \begin{bmatrix} C_n \times (C_j - 1) & C_n - 1 & 1 \\ C_n & 1 - C_n & -1 \\ 0 & C_j & -C_j \end{bmatrix} \times \begin{bmatrix} C_i - 1 \\ C_m - 1 \\ -1 \end{bmatrix}$$

Unfortunately, this describes a non-causal system with respect to line $\{i, j\}$. In practice, you attend to line $\{m, m\}$ only when necessary, since the outcome of $\{i, j\}$ deduces whether further betting needs to occur. The idea is to form a hedge for the first line $\{i, j\}$, and if the hedge line wins, then form a hedge for the second line - an inescapable dependency.

We simplify this situation through an assumption that $C_m, C_n \in \{C_i, C_j\}$. All that needs to be understood is how to structure C_i, C_j, C_m, C_n in general terms to maximize the unit PnL ($\beta = 1$):

$$\text{PnL} = \frac{C_n \times (C_i - 1) \times (C_j - 1) + (C_m - 1) \times (C_n - 1) - 1}{C_j \times C_n} \tag{13}$$

We isolate multipliers according to lines as:

$$f(C_x, C_y) = \frac{(C_x - 1) \times (C_y - 1)}{C_y} \tag{14}$$

$$\text{PnL} = f(C_i, C_j) + \frac{1}{C_j} \times (f(C_m, C_n) - \frac{1}{C_n}) \tag{15}$$

But we are reminded that Equation 1 constrains C_i and C_j to V:

$$\text{Eq 1} \longrightarrow f(C_x, V) = \frac{(C_x - 1) \times (C_x - V \times C_x + 1)}{C_x} \tag{16}$$

$$\frac{\partial f}{\partial C_x} = 0 \longrightarrow C_x = \frac{1}{\sqrt{V - 1}}; \quad C_y = \frac{1}{V - \sqrt{V - 1}} \quad \text{where } V > 1 \tag{17}$$

Empirically, V tends to be in the space of $1 < V < 1.05$, so the promotional line should always choose the underdog; $C_i >> C_j$ and $C_i = C_U, C_j = C_F\}$. An identical analysis can be extrapolated for $\{C_m, C_n\}$, since both functions $f(\cdot, \cdot)$ define the PnL.

1.3.2 Bonus Bet

Most analysis is left out as an exercise for the reader, since there is a singular term of interest which requires modification. We rewrite the PnL equation as such:

$$\text{PnL} = \frac{C_n \times (C_i - 1) \times (C_j - 1) + (C_m - 2) \times (C_n - 1) - 1}{C_j \times C_n} \tag{18}$$

$$\text{PnL} = (\text{PnL from Eq 13}) - \frac{C_n - 1}{C_j \times C_n} \tag{19}$$

An additional negative term implies a lower PnL than what is observed with a site credit as an insurance payoff. The promotional line should still choose the underdog, and hedge with the favourite to maximize the PnL.

1.3.3 Expected Conversion Rates

Without loss of generality, we assume lines $\{i, j\}$ and $\{m, n\}$ will be identical. PnL for the site credit case is rewritten in terms of C_i and V:

$$\text{Conv \%} = \text{Unit PnL} = 2V - (C_i \times (V^2 - 1) + \frac{1}{C_i} + 1) \tag{20}$$

$$\frac{\partial}{\partial C_x} \text{PnL} = 0 \longrightarrow C_x^{'} = \frac{1}{\sqrt{V^2 - 1}}; \quad \text{PnL}\Big|_{C_x = C_x^{'}} = 2V - 2\sqrt{V^2 - 1} - 1 \tag{21}$$

Not exactly a fair fight. Conversion grows linearly with vig, but reduces quadratically. There is a broad range of conversions; a low vig enables enables a Conv % $> 70\%$, a rare find. Tight lines punish you, with conversion in the range of $50\% < x < 65\%$, with more fruitful conversions sourced from bankrolls with greater excess liquidity.

For the bonus bet case, we mask certain negligible terms as R:

$$\text{Conv }\% = \text{Unit PnL} = V \times (\frac{C_i - 2}{C_i}) - (V^2 - 1) \times (C_i - 1) + R \tag{22}$$

$$\frac{\partial}{\partial C_x}\text{PnL} = 0 \longrightarrow C_x^{'} = \sqrt{\frac{2V}{V^2 - 1}}; \quad \text{PnL}\Big|_{C_x = C_x^{'}} = V^2 + V - 2\sqrt{2}\sqrt{V^3 - V} - 1 \tag{23}$$

Evidently, this PnL has a smaller positive bias than the site credit case, and decreases quadratically with respect to vig. Oppositely to the site credit, this bias grows proportional to $\frac{C_i - 2}{C_i}$, so this PnL is more right-skewed to the site credit PnL; A suitable conversion rate requires $C_i >> 2$ and ranges anywhere from $30\% < x < 50\%$.

The morale of the story reveals that vigorish is the killer in PnL. It's better to wait and find a line with as little vig as possible to maximize gains. With bonus bets, you can recoup any losses with vigorish by upsizing your hedges; the question remaining is a personal one - can you afford it?

1.3.4 Updating Hedges

This section aims to bring a causal interpretation to the non-causal derivation for hedges. Upon discovery of a suitable line, make an assumption that E_1 will follow E_0 correspondingly; i,e $\{i, j\} = \{m, n\} \longrightarrow \{C_i, C_j\} = \{C_m, C_n\}$. If E_1 is far from E_0, then h_1 and theoretical PnL will diverge from our calculation.

We restate that our goal is maximize PnL, and finding the intersecting PnL across case scenarios simplifies this task. However, it is unnecessary to do so, and is highly unrealistic. To find the maximal PnL across PnL_2 and PnL_3, we simply use the greedy approach described in Section 1.2. States are independent, so once E_0 is realized, we have incurred a loss, but that loss was contained based on E_0. Afterwards, we play a game of catch-up, where we try to convert the insurance payoff maximally to offset this loss.

1.4 Epilogue

This section is closed for further analysis. All sport promotions follow the structure described, and conversions have no risk for ruin.

2 Casino

2.1 Prologue

Casino promotions have been previously considered unbreakable - unlike sportsbook promotions, which are well-documented, with a fully hedged risk profile, casino promotions are a beast that has not been thoroughly explored. This is because casino promotions have a stochastic component that is missing from sportsbooks. Converting sportsbook promotions is a simple ordeal - there is a finite amount of possible outcomes, where an optimal strategy can be observed above through linear algebra.

Determining an optimal strategy for casino promotion conversion is tricky, because the amount of states to keep under consideration is proportional to your strategy. There is a large range of strategies where your PnL goes negative, and forces a ruin. We aim to explore methods that saddle risk and PnL.

2.2 Games of Interest

Baccarat and Roulette are games that are open to hedged positions. Baccarat and Roulette both employ 2-way hedges differently. For now, assume all bets are staked with cash.

2.2.1 Baccarat

We define 2 separate PnL states, as T_u and T_v, where the combining PnL state is $T = T_u + T_v$. Bets are sized as h_u and h_v for each state. Baccarat has 2 mutually exclusive states, Player and Banker, so state

u always bets on the opposing side of state v. We define a random variable $X \in \{P, B\}$, where u and v take Player and Banker states respectively. This defines our PnL state after one iteration of play:

$$T_u = h_u \times \mathbb{I}(X = P) - h_u \times \mathbb{I}(X = B) \tag{24}$$

$$T_v = \frac{19}{20} h_v \times \mathbb{I}(X = B) - h_v \times \mathbb{I}(X = P) \tag{25}$$

$$T = (h_u - h_v) \times \mathbb{I}(X = P) + (\frac{19}{20} h_v - h_u) \times \mathbb{I}(X = B) \tag{26}$$

Given $\Pr(X = P) \approx \Pr(X = B)$, $\mathbb{E}_X[T] \approx -\frac{1}{40} h_v$. This is where the house edge, or vigorish is numerically presented. You are expected to lose an amount proportional to h_v on each iteration. In practice, $\Pr(X = P) = \Pr(X = B) - \epsilon$, where ϵ is a small probability.

There is no application in hedging baccarat positions to grow your bankroll. Merit lies in the PnL state transition. In the case of Player or Banker, a wealth transfer was completed. If $\mathbb{I}(X = P) = 1$, then $T_u = +h_u$ and $T_v = -h_v$. Oppositely, if $\mathbb{I}(X = B) = 1$, then $T_u = -h_u$ and $T_v = +\frac{19}{20} h_v$. On average, we pay a small commission to transfer funds across states proportional to h_u and h_v.

Unfortunately, this transfer is stochastic/non-deterministic, with a negative expected value. It's best to not play at all. However, we can define an optimal hedge if h_u or h_v are forced to be nonzero. A misconception is that an optimal hedge **maximizes** $\mathbb{E}_X[T]$. An optimal hedge is one that **minimizes** losses; in certain instances, these ideas are not simply interchangeable. We reframe our optimization to size hedges accordingly:

$$\mathbb{L} = \min(\Pr(X = P) \times (h_u - h_v),\ \Pr(X = B) \times (\frac{19}{20} h_v - h_u)) \tag{27}$$

$$\arg\max_{h_u, h_v} \mathbb{L} := \{h_u \approx \frac{39}{40} h_v\} \longrightarrow T_{X=P} = T_{X=B} = -\frac{1}{40} h_v \tag{28}$$

Each iteration incurs a loss, but it is contained. An alternative strategy exploits a skewed strategy, where one state incurs a loss double to this, where the other state faces no loss. Reason for presentation is explained in further sections:

$$\{h_u = h_v\} \longrightarrow T_{X=P} = 0; \quad T_{X=B} = -\frac{1}{20} h_u \tag{29}$$

2.2.2 Roulette

In roulette, you wager on a number from 37 numbers, each with a payout of 35:1 proportional to your stake. State u stakes on the first n numbers, and state v stakes on the next $37 - n$ numbers $\longrightarrow$ a fully hedged position, since both positions are mutually exclusive. The random variable is defined as $X \in \{0, 1, 2, ..., 36\}$, with $\Pr(X = x) = \frac{1}{37}$. PnL after one iteration is defined as:

$$T_u = (36 - n) \times h_u \times \mathbb{I}(X < n) - n \times h_u \times \mathbb{I}(X \geq n) \tag{30}$$

$$T_v = (n - 1) \times h_v \times \mathbb{I}(X \geq n) - (37 - n) \times h_v \times \mathbb{I}(X < n) \tag{31}$$

$$T = ((36 - n) \times (h_u - h_v) - h_v) \times \mathbb{I}(X < n) \tag{32}$$

$$+ ((n - 1) \times (h_v - h_u) - h_u) \times \mathbb{I}(X \geq n) \tag{33}$$

The dynamic with roulette is that there is control over the likelihood of a wealth transfer from one state to another. In other words, $\Pr(X < n) = \frac{n}{37}$ and reversely, $\Pr(X \geq n) = \frac{37-n}{37}$. We normalize $T_{X \geq n}$ to the value on transfer, and optimize:

$$T_{X<n} = 0 \longrightarrow \{h_u = h_v \times \frac{37 - n}{36 - n}\} \longrightarrow T_{X \geq n} = -h_v \times \frac{36}{36 - n} \iff \hat{T}_{X \geq n} = \frac{T_{X \geq n}}{(n - 1) \times h_v} \tag{34}$$

$$\mathbb{E}_X[\hat{T}_{X \geq n}] = \Pr(X \geq n) \times \hat{T}_{X \geq n} \longrightarrow \arg\max_n \mathbb{E}_X[\hat{T}_{X \geq n}] := \{n = 6\} \tag{35}$$

We define $\mathbb{H}_u$ and $\mathbb{H}_v$ as the aggregates of states u and v to instate isomorphism across roulette and baccarat; i.e, $\mathbb{H}_u = n \times h_u$ and $\mathbb{H}_v = (37 - n) \times h_v$. According to n, we list the change in position under each case:

$$T_u = \begin{cases} +\mathbb{H}_u \times \frac{36-n}{n} & X < n \\ -\mathbb{H}_u & X \geq n \end{cases} \tag{36}$$

$$T_v = \begin{cases} -\mathbb{H}_v & X < n \\ +\mathbb{H}_v \times \frac{n-1}{37-n} & X \geq n \end{cases} \tag{37}$$

$$\{\mathbb{H}_u = \mathbb{H}_v \times \frac{n}{36-n}\} \longrightarrow T_{X<n} = 0 \iff T_{X\geq n} = -\mathbb{H}_v \times \frac{36}{(37-n)\times(36-n)} \tag{38}$$

2.3 Promotion Variants

The structure of specific promotions is described, with every parameter of interest.

2.3.1 Restricted Deposit Match

An initial deposit subscribes to an addition of bonus capital, at an arbitrary rate R_m proportional to the deposit. The entire balance is then restricted as bonus capital. In other words, a cash deposit ϕ provides a restricted position of $\phi \times (R_m + 1)$; this is interchangeable with leverage, where we have the ability to utilize more capital than invested.

Unfortunately, bonus capital cannot simply be withdrawn, and has an arbitrary playthrough rate p_t. The value of playthrough is proportional to the additive bonus capital. Therefore, in order to convert the bonus capital to cash, the casino impels a wagered amount of $\phi \times p_t \times R_m$.

A high playthrough implies a longer duration of play, which becomes susceptible to the house edge; Even an impossible game with positive expectation is susceptible to gambler's ruin. For these reasons, an act of degenerate betting is fruitless due to volatility and non-determinism. It's especially dangerous since the balance is restricted - there exists a large sample space where you lose the bonus capital, inclusive of your initial deposit.

2.3.2 Unrestricted Deposit Match

This is a simpler case of Section 2.3.1, where the cash deposit ϕ is unrestricted and independent of the total bonus balance. There exists 2 balances, a cash balance of ϕ, and a bonus balance of $\phi \times R_m$.

2.3.3 Insured Loss

A net loss of L is insured, and reimbursed through site credit or cash, with defined proportion α. This is a generalized variation of an insured loss on a sportsbet. Sportsbooks only insure the first bet, wheras casino insurance exists for any sequence of bets that is eventually absorbed as a net loss. We will formulate a reduction from Section 2.3.2 to this promotion variant.

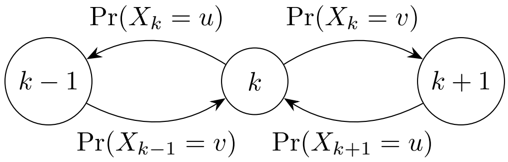
2.4 Generalization of Hedge States

Previous sections described a 1-state prognosis of wealth transfers. This section aims to generalize it to n-states.

2.4.1 Biased Random Walk

Each state is associated with a node, which recognizes the PnLs of mutually exclusive states $\{u, v\}$. Transitions are abstracted as steps in a biased random walk; a step away from the current state implies a new pair of PnL states. These states must be memoryless and invariant of the path taken to realize this state. To achieve this, hedges are sized accordingly prior to the transition.

A Markov chain of n states can be built through layers of a 3-state component. Recursively apply this component to the tail states of the current chain to generate the next chain. X_k defines a random variable where $X_k \in \{u, v\}$. Without loss of generality, $X_k = u$ implies a wealth transfer from v to u, so a step in the left $(k-1)$ or right $(k+1)$ direction is taken from current state k:



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To preserve the Markov property, hedges are geometrically structured based on the distance from the origin state (k) to the current state (i):

$$\mathbb{H}^i = \begin{bmatrix} \mathcal{H}_u^i \\ \mathcal{H}_v^i \end{bmatrix} = \begin{bmatrix} (C_u)^{k-i} \times h_u \\ (C_v)^{i-k} \times h_v \end{bmatrix} \tag{40}$$

where C_u, C_v denote the stake multipliers on an arbitrary hedge. This hedge profile defines a memoryless PnL state at each step:

$$\mathbb{T}^i = \begin{bmatrix} \mathcal{T}_u^i \\ \mathcal{T}_v^i \end{bmatrix} = \begin{bmatrix} h_u \\ h_v \end{bmatrix} \otimes \begin{bmatrix} f_u(k-i) \\ f_v(i-k) \end{bmatrix} + \begin{bmatrix} \mathcal{T}_u^k \\ \mathcal{T}_v^k \end{bmatrix} \tag{41}$$

$$f_\alpha(n) = \sum_{j=1}^{n} (C_\alpha)^j - \sum_{j=n+1}^{0} (C_\alpha)^j = \begin{cases} n & C_\alpha = 1 \\ [(C_\alpha)^n - 1] \times \frac{C_\alpha}{C_\alpha - 1} \times \text{sgn}(n) & C_\alpha \neq 1 \end{cases} \tag{42}$$

2.4.1.1 Absorption States

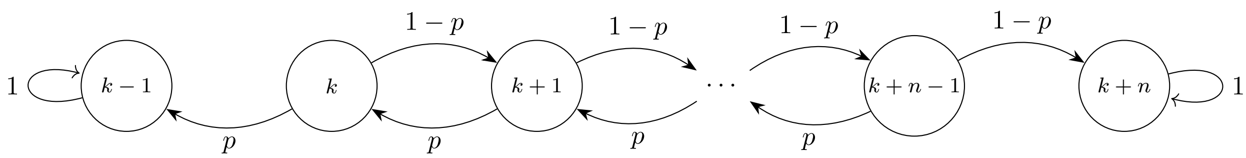
This section explores possible absorption states, where the bonus balance is converted to cash:

- **Soft** Conversion $\longrightarrow$ A loss on the bonus balance $\iff$ A win on the hedge balance
- **Hard** Conversion $\longrightarrow$ The bonus balance is wagered until playthrough is met $\iff$ $\phi \times p_t \times R_m$

These conversions are of little value when considered independently. Soft conversions do not account for sequences where the bonus balance rarely incurs a loss; an open risk profile for ruin. Hard conversions are stochastic and volatile in nature. The expected conversion is proportional to $\phi \times p_t \times R_m$, leaving a negative PnL in average cases. This can be derived through an application of **CLT** (Central Limit Theorem).

However, when used in conjunction, the pitfalls of one conversion method are mitigated by the strengths of the other. A hard conversion should only be made if the bonus capital margin is great enough for a non-negative conversion with respect to PnL $\longrightarrow$ A sequence of bonus balance wins grows this margin, with a correctly corresponding hedge profile.

These conversion states are mapped through absorption states in an n-state Markov chain with an origin state at step k. Without loss of generality, state u is our hedge side and state v is our promotional side. Absorption states are at $\{i_u = k-1, i_v = k+n\}$. For brevity, $\Pr(X_i = u) = p$ and $\Pr(X_i = v) = 1-p$:



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Note that P is our transition matrix for the Markov chain, which is decomposed into its transient and absorbing submatrices, which are $\{Q, R\}$ respectively. P^j defines the probability of being at a certain state after j steps. We aim to find the absorbing probabilities of each absorbing state:

$$\lim_{j\to\infty} P^j = \lim_{j\to\infty} \begin{bmatrix} Q^j & (I_t - Q^j)(I_t - Q)^{-1}R \\ \mathbf{0}_{r\times t} & I_r \end{bmatrix} = (I_t - Q)^{-1}R = B \tag{44}$$

$$B = \begin{bmatrix} 1 & p-1 & 0 \\ -p & 1 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix}^{-1} \times \begin{bmatrix} p & \mathbf{0}_{t-1} \\ \mathbf{0}_{t-1} & 1-p \end{bmatrix} \tag{45}$$

$I_t - Q$ is a tridiagonal Toeplitz matrix with a closed-form inversion that is best defined recursively. Redefining $A(t) = (I_t - Q(t))^{-1}$ for t transient states, the determinant is computed as

$$\det(A(t)) = \begin{cases} \det(A(t-1)) + (p^2 - p) \times \det(A(t-2)) & \forall t = 3, 4, \dots \\ p^2 - p + 1 & t = 2 \\ 1 & t = 1 \end{cases} \tag{46}$$

Our values of interest are $A_{0,0}(t)$ and $A_{0,t-1}$(t), so

$$\begin{bmatrix} A_{0,0}(t) \\ A_{0,t-1}(t) \end{bmatrix} = \frac{1}{\det(A(t))} \begin{bmatrix} \det(A(t-1)) \\ (1-p)^{t-1} \end{bmatrix} \tag{47}$$

Therefore, the steady-state probabilities of absorption at either state given origin state k are marked by $B_k(t)$:

$$B_k^T(t) = \begin{bmatrix} A_{0,0}(t) \times p \\ A_{0,t-1}(t) \times (1-p) \end{bmatrix} \tag{48}$$

2.4.1.2 Optimal Number of States

An optimal stopping point is where our margin growth stagnates between the bonus balance and the hedge balance. This can be rationalized as a greedy algorithm, where we halt at a state n if transitioning to the next state $(n+1)$ yields a worse outcome for the statewise PnL.

Initially, we write this as a generalized form with the inclusion of a table limit, $h_{\max}$, which represents the maximum possible stake on one iteration of play:

$$\hat{n} = 1 + \max\{i \mid C_v \times \mathcal{H}_v^i > \mathcal{H}_u^i \quad \forall i = k, k+1, \dots, k + i_{\max}\} \tag{49}$$

$$i_{\max} = \max\{i \mid \mathcal{H}_u^i \leq h_{\max} \cap \mathcal{H}_v^i \leq h_{\max} \quad \forall i = k, k+1, \dots\} \tag{50}$$

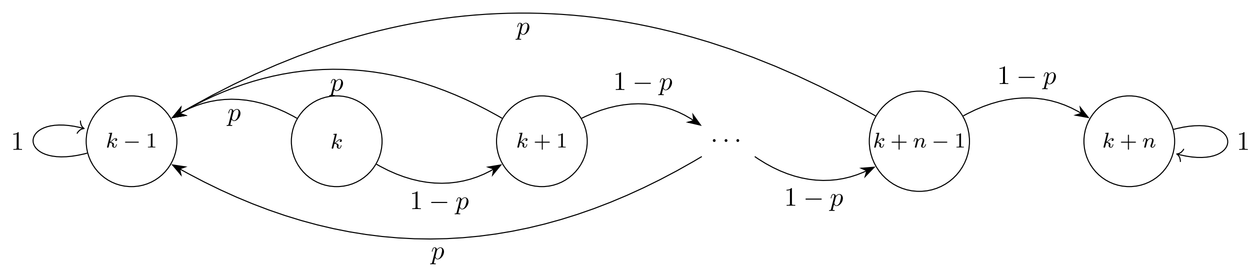
This is simplified for the biased random walk:

$$\hat{n} = \min\{\lceil \frac{\ln(h_u) - \ln(h_v) - \ln(C_v)}{\ln(C_u) + \ln(C_v)} \rceil,\ 1 + i_{\max}\} \tag{51}$$

$$i_{\max} = \begin{cases} \min\{\frac{\ln(h_u) - \ln(h_{\max})}{\ln(C_u)}, \frac{\ln(h_{\max}) - \ln(h_v)}{\ln(C_v)}\} & C_u < 1 \cap C_v \geq 1 \\ \infty & C_u \geq 1 \cap C_v < 1 \end{cases} \tag{52}$$

2.4.2 Asymmetric Random Walk

We aim to map the martingale betting system to a mathematical domain. This process is asymmetric, since steps at each state are nonuniform. This method attempts to maximize the likelihood of a soft conversion, with rationale that each state is simply one variable step away from a soft absorption. To satisfy this idea, assume the promotional side stakes its entire bankroll at each step.



(53)

$$\begin{bmatrix} \mathcal{H}_v^i \\ \mathcal{T}_v^i \end{bmatrix} = h_v \begin{bmatrix} (C_v + 1)^{i-k} \\ (C_v + 1)^{i-k} - 1 \end{bmatrix} \quad \forall i = k, k+1, \dots, k + \hat{n}$$

(54)

2.4.2.1 Deriving $\mathcal{H}_u^i$

State definitions for u were left as variable. Defining hedge states optimally requires a non-negative soft conversion, and a growing margin for the hard conversion:

$$\mathcal{H}_v^i \times C_v \geq \mathcal{H}_u^i \quad \cap \quad \mathcal{H}_u^i \times C_u \geq |\mathcal{T}_u^i| \tag{55}$$

$$\text{where } \mathcal{T}_u^i = T_u^k - \sum_{j=k+1}^{i} \mathcal{H}_u^{j-1} \quad \forall i = k, k+1, \ldots, k+\hat{n} \tag{56}$$

This is restated in the standard form of linear programming, with a disregarded dummy objective. Assume $\mathcal{H}_u$ is a vector of hedges for all i, with a defined cardinality, $\|\mathcal{H}_u\| = \hat{n}$,

$$\textbf{Find vector } \mathcal{H}_u \tag{57}$$

$$\textbf{subject to } A^{-1}C \leq \mathcal{H}_u \tag{58}$$

$$\textbf{and } \mathcal{H}_u \leq B \tag{59}$$

with definitions for A, B, C as follows:

$$A = \begin{bmatrix} C_u & 0 & 0 \\ -1 & C_u & \ddots \\ -1 & \ddots & \ddots \end{bmatrix} B = \begin{bmatrix} \min\{h_v \times C_v, h_{\max}\} \\ \vdots \\ \min\{h_v \times C_v(C_v+1)^{\hat{n}-1}, h_{\max}\} \end{bmatrix} C = \begin{bmatrix} P - \mathcal{T}_u^k \\ \vdots \\ P - \mathcal{T}_u^k \end{bmatrix} \tag{60}$$

$$\text{where } \mathcal{H}_u^k \times C_u = P - \mathcal{T}_u^k \ \mid \ -\mathcal{T}_u^k \leq P - \mathcal{T}_u^k \leq h_v \times C_u C_v \tag{61}$$

These definitions appear to be self-referential. This is subverted through an assumption that P is given, subject to constraints, which is the lower bound for profit obtained from a soft conversion from any state. Solving $A^{-1}C$ yields a closed-form for the first inequality:

$$A^{-1}C = (P - \mathcal{T}_u^k) \times (C_u)^{-1} \begin{bmatrix} 1 \\ \vdots \\ [(C_u)^{-1} + 1]^{\hat{n}-1} \end{bmatrix} \leq \begin{bmatrix} \mathcal{H}_u^k \\ \vdots \\ \mathcal{H}_u^{k+(\hat{n}-1)} \end{bmatrix} \tag{62}$$

2.4.2.2 Optimal Number of States

We derive the maximal n that satisfies table limits, according to the givens:

$$\hat{n} = 1 + i_{\max} = 1 + \min\{\lfloor \frac{\ln(h_{\max}) + \ln(C_u) - \ln(P - \mathcal{T}_u^k)}{\ln(C_u+1) - \ln(C_u)} \rfloor, \lfloor \frac{\ln(h_{\max}) - \ln(h_v)}{\ln(C_v+1)} \rfloor\} \tag{63}$$

2.4.2.3 Steady-State Probabilities

There is strictly one path to reach state $k + \hat{n}$. Steady-state probabilities are trivial to calculate:

$$B_k^T = \begin{bmatrix} 1 - (1-p)^{\hat{n}} \\ (1-p)^{\hat{n}} \end{bmatrix} \tag{64}$$

2.5 Logistics of a Hard Conversion

We first identify the margin between PnL states. Recall that at state $\hat{n}$, $\mathcal{T}_v^{\hat{n}} > 0$ and $\mathcal{T}_u^{\hat{n}} < 0$ where $|\cdot|_1$ denotes an L1-norm:

$$\triangle T : |\mathbb{T}^{\hat{n}}|_1 = \mathcal{T}_u^{\hat{n}} + \mathcal{T}_v^{\hat{n}} \geq 0 \tag{65}$$

The balance on state v must be played through an amount of $\phi \times p_t \times R_m$ to yield a cash conversion. One viable option is slots, which offers a specific Return-To-Player (RTP) on individual spins. RTP is an abstraction of the expected value for each spin, proportional to some given stake. The idea is to play with small stakes, so that there is application from CLT. In some sense, the empirical RTP will converge to the theoretical RTP.

S_i is defined as a random variable from the slots distribution, where $\mathbb{E}[S_i] = u$ and $\text{Var}[S_i] = \sigma^2$. We formulate a summation of these **iid** variables, according to a prefixed stake b:

$$\bar{n} = \frac{M}{b} = \frac{\phi \times p_t \times R_m}{b} \quad \bar{S}_n = b\sum_{i=1}^{n} S_i \approx \mathcal{N}(nbu, nb^2\sigma^2) \longrightarrow \bar{S}_n \approx \mathcal{N}(Mu, Mb\sigma^2) \tag{66}$$

We normalize the Gaussian approximation according to M:

$$Z = \frac{\bar{S}_n}{M} \approx \mathcal{N}(u, \frac{\sigma^2}{\bar{n}}) \longrightarrow R = \Pr(Z \leq 1 - \frac{\triangle T}{M}) \tag{67}$$

Empirically, $\{u, \sigma^2\}$ fall in ranges of $[0.94, 0.97]$ and $[6, 21]$ respectively. R is the probability of an observed sequence that generates a poor conversion, and causes a negative PnL. This is our quantified risk profile.

2.6 Application

This section describes the application of either random walk to the promotional variants. We inscribe a mapping from the problem description to initial PnL and hedge states. Initial states for state u are left for fine-tuning according to a desired risk profile.

Simply add the initialized states to previously defined states, according to the state step in the chain.

2.6.1 Restricted Deposit Match

$$\begin{bmatrix} \mathcal{T}_u^k \\ \mathcal{T}_v^k \end{bmatrix} = \begin{bmatrix} -\phi \\ \phi \times (R_m + 1) \end{bmatrix} \quad h_v = \phi \times (R_m + 1) \tag{68}$$

2.6.2 Unrestricted Deposit Match

$$\begin{bmatrix} \mathcal{T}_u^k \\ \mathcal{T}_v^k \end{bmatrix} = \begin{bmatrix} 0 \\ \phi \times R_m \end{bmatrix} \quad h_v = \phi \times R_m \tag{69}$$

2.6.3 Insured Loss

$$\begin{bmatrix} \mathcal{T}_u^{k-1} \\ \mathcal{T}_v^{k-1} \end{bmatrix} \leftarrow \begin{bmatrix} \mathcal{T}_u^{k-1} \\ \mathcal{T}_v^{k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha L \end{bmatrix} \quad \begin{bmatrix} \mathcal{T}_u^k \\ \mathcal{T}_v^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad h_v = L \tag{70}$$

2.7 Epilogue

This section is open for further analysis, because of an open risk profile.